UNIT -2-ALGEBRA-10 MARKS

Chapter 03-04

ALGEBRA OF MATRICES & DETERMINANTS

If I have the belief that I can do it, I will acquire all the capacity to do it even if I may not have it at the beginning!

BASIC ALGEBRA OF MATRICES

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Matrix - a basic introduction: A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as *elements* or the *entries* of the matrix. It is denoted by the **upper case letters** *i.e.* A, B, C etc.

Consider a matrix A given as, $$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}.$$

Here in matrix A depicted above, the horizontal lines of elements are said to constitute ***rows*** of the matrix A and vertical lines of elements are said to constitute ***columns*** of the matrix. Thus matrix A has ***m* rows** and ***n* columns**. The array is enclosed by brackets $\left[\ \right]$, the parentheses $(\ )$ and the double vertical bars $\|\ \|$.

> ➢ *A matrix having m rows and n columns is called a matrix of order* $m\times n$ *(read as '**m by n**' matrix). And a matrix A of order* $m\times n$ *is depicted as* $A=\left[a_{ij}\right]_{m\times n}$; $i,j\in\mathbb{N}$.
>
> ➢ *Also in general,* a_{ij} *means an element lying in the* i^{th} *row and* j^{th} *column.*
>
> ➢ *No. of elements in the matrix* $A=\left[a_{ij}\right]_{m\times n}$ *is given as* $(m)(n)$.

02. Types of Matrices:

a) Column matrix: A matrix having only one column is called a *column matrix* or *column vector*.

e.g. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}, \begin{bmatrix} 8 \\ 5 \end{bmatrix}_{2\times 1}$.

➲ **General notation:** $A=\left[a_{ij}\right]_{m\times 1}$.

b) Row matrix: A matrix having only one row is called a *row matrix* or *row vector*.

e.g. $\begin{bmatrix} -1 & 2 & \sqrt{3} & 4 \end{bmatrix}_{1\times 4}, \begin{bmatrix} 2 & 5 & 0 \end{bmatrix}_{1\times 3}$

➲ **General notation:** $A=\left[a_{ij}\right]_{1\times n}$.

c) Square matrix: It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $m\times n$ matrix is said to constitute a square matrix if $m=n$ and is known as a ***square matrix of order 'n'***.

e.g. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3.

➲ **General notation:** $\mathrm{A}=\left[a_{ij}\right]_{n\times n}$.

d) Diagonal matrix: A square matrix $\mathrm{A}=\left[a_{ij}\right]_{m\times m}$ is said to be a *diagonal matrix* if $a_{ij}=0$, when $i\neq j$ *i.e.,* all its non- diagonal elements are *zero.*

e.g. $\begin{bmatrix}2&0&0\\0&5&0\\0&0&4\end{bmatrix}_{3\times 3}$ is a diagonal matrix of order 3.

> ➢ *Also there is* ***one more notation*** *specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can be also written as* $diag\begin{pmatrix}2&5&4\end{pmatrix}$.
>
> ➢ *Note that the elements* $a_{11},a_{22},a_{33},\ldots,a_{mm}$ *of a square matrix* $A=\left[a_{ij}\right]_{m\times n}$ *of* ***order m*** *are said to constitute the* ***principal diagonal*** *or simply* ***the diagonal of the square matrix A****. And these elements are known as* ***diagonal elements of matrix A****.*

e) Scalar matrix: A *diagonal matrix* $\mathrm{A}=\left[a_{ij}\right]_{m\times m}$ is said to be a *scalar matrix* if its *diagonal elements* are **equal** *i.e.*, $a_{ij}=\begin{cases}0, & \text{when } i\neq j\\ k, & \text{when } i=j \text{ for some constant } k\end{cases}$.

e.g. $\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}_{3\times 3}$ is a scalar matrix of order 3.

f) Unit or Identity matrix: A square matrix $\mathrm{A}=\left[a_{ij}\right]_{m\times n}$ is said to be an *identity matrix* if $a_{ij}=\begin{cases}1, \text{ if } & i=j\\ 0, \text{ if } & i\neq j\end{cases}$.

A *unit matrix* can also be defined as the *scalar matrix* each of whose diagonal elements is *unity*. We denote the identity matrix of order m by I_m or I.

e.g. $\mathrm{I}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\mathrm{I}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

g) Zero matrix or Null matrix: A matrix is said to be a *zero matrix* or *null matrix* if each of its elements is '0'.

e.g. $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\end{bmatrix}$.

h) Horizontal matrix: A $m\times n$ matrix is said to be a *horizontal matrix* if $m<n$.

e.g. $\begin{bmatrix}1&2&0\\5&4&7\end{bmatrix}_{2\times 3}$.

i) Vertical matrix: A $m\times n$ matrix is said to be a *vertical matrix* if $m>n$.

e.g. $\begin{bmatrix}2&5\\0&7\\3&1\end{bmatrix}_{3\times 2}$.

j) Triangular matrix:

Lower triangular matrix: A square matrix is called a lower triangular matrix if $a_{ij}=0$ *when* $i<j$.

e.g. $\begin{bmatrix}1&0&0\\2&2&0\\0&5&3\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&0&0\\0&5&0\end{bmatrix}$, $\begin{bmatrix}2&0&0\\3&2&0\\4&5&7\end{bmatrix}$.

Upper triangular matrix: A square matrix is called an upper triangular matrix if $a_{ij}=0$ *when* $i>j$.

e.g. $\begin{bmatrix}1&2&4\\0&5&8\\0&0&3\end{bmatrix}$, $\begin{bmatrix}1&3&9\\0&0&0\\0&0&5\end{bmatrix}$.

03. Equality of Matrices: Two matrices A and B are said to be equal and written as A = B, if they are of the **same orders** and their **corresponding elements are identical** *i.e.* $a_{ij} = b_{ij}$ for all *i* and *j*. That is $a_{11} = b_{11}, a_{22} = b_{22}, a_{32} = b_{32}$ etc.

04. Addition of matrix: If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and
B and is denoted by 'A + B'.
Thus if $\mathrm{A} = \left[a_{ij}\right], \mathrm{B} = \left[b_{ij}\right] \Rightarrow \mathrm{A} + \mathrm{B} = \left[a_{ij} + b_{ij}\right]$.

➲ **Properties of matrix addition:**

- *Commutative property:* $\mathrm{A} + \mathrm{B} = \mathrm{B} + \mathrm{A}$
- *Associative property:* $\mathrm{A} + (\mathrm{B} + \mathrm{C}) = (\mathrm{A} + \mathrm{B}) + \mathrm{C}$
- *Cancellation laws:* ***i)*** *Left cancellation* - $\mathrm{A} + \mathrm{B} = \mathrm{A} + \mathrm{C} \Rightarrow \mathrm{B} = \mathrm{C}$
  ii) *Right cancellation* - $\mathrm{B} + \mathrm{A} = \mathrm{C} + \mathrm{A} \Rightarrow \mathrm{B} = \mathrm{C}$.

05. Multiplication of a matrix by a scalar: If an $m \times n$ matrix A is multiplied by a scalar *k* (say), then the new *k*A matrix is obtained by multiplying each element of matrix A by scalar *k*. Thus if $\mathrm{A} = \left[a_{ij}\right]$ and it is multiplied by a scalar *k* then, $k\,\mathrm{A} = \left[k\,a_{ij}\right]$, *i.e.,* $\mathrm{A} = \left[a_{ij}\right] \Rightarrow k\,\mathrm{A} = \left[k\,a_{ij}\right]$.

e.g. $\mathrm{A} = \begin{bmatrix} 2 & -1 \\ 6 & 4 \end{bmatrix} \Rightarrow 3\mathrm{A} = \begin{bmatrix} 6 & -3 \\ 18 & 12 \end{bmatrix}$.

06. Multiplication of two matrices: Let $\mathrm{A} = \left[a_{ij}\right]$ be a $m \times n$ matrix and $\mathrm{B} = \left[b_{jk}\right]$ be a $n \times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m \times p$ matrix $\mathrm{C} = \left[c_{ik}\right]$ such that $\mathrm{C}_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB *i.e.* "C = AB'.

[For better illustration, you need to follow a few examples.]

➲ **Properties of matrix multiplication:**

- Note that the **product AB is defined *only* when** the number of columns in matrix A is equal to the number of rows in matrix B.
- If A and B are $m \times n$ and $n \times p$ matrices respectively then the matrix AB will be an $m \times p$ matrix *i.e.,* order of matrix AB will be $m \times p$.
- In the product AB, A is called the ***pre-factor*** and B is called the ***post-factor***.
- If two matrices A and B are such that AB is possible then it is ***not necessary*** that the product BA is also possible.
- If A is a $m \times n$ matrix and both AB as well as BA are defined then B will be a $n \times m$ matrix.
- If A is a $n \times n$ matrix and I_n be the unit matrix of order n then, $\mathrm{A}\,\mathrm{I}_n = \mathrm{I}_n\,\mathrm{A} = \mathrm{A}$.
- Matrix multiplication is ***associative*** *i.e.*, $\mathrm{A}(\mathrm{BC}) = (\mathrm{AB})\mathrm{C}$.
- Matrix multiplication is ***distributive over the addition*** *i.e.*, $\mathrm{A}(\mathrm{B} + \mathrm{C}) = \mathrm{AB} + \mathrm{AC}$.

➲ <u>**Idempotent matrix:**</u> A square matrix A is said to be an idempotent matrix if $\mathbf{A}^2 = \mathbf{A}$.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

07. Transpose of a Matrix: If $A=\left[a_{ij}\right]_{m\times n}$ be an $m\times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a ***transpose of matrix A***. The transpose of A is denoted by A' or A^T or A^c *i.e.,* if $A=\left[a_{ij}\right]_{m\times n}$ then, $A^T=\left[a_{ji}\right]_{n\times m}$.

For example, $\begin{bmatrix}3 & 2 & 0\\ 1 & -2 & 6\end{bmatrix}^T=\begin{bmatrix}3 & 1\\ 2 & -2\\ 0 & 6\end{bmatrix}$.

➲ **Properties of Transpose of matrices:**

- $(A+B)^T=A^T+B^T$
- $(A-B)^T=A^T-B^T$
- $(A^T)^T=A$
- $(kA)^T=kA^T$ where, k is a constant
- $(AB)^T=B^TA^T$
- $(ABC)^T=C^TB^TA^T$

08. Symmetric matrix: A square matrix $A=\left[a_{ij}\right]$ is said to be a *symmetric matrix* if $A^T=A$.

That is, if $A=\left[a_{ij}\right]$ then, $A^T=\left[a_{ji}\right]=\left[a_{ij}\right]\Rightarrow A^T=A$.

For example: $\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix},\begin{bmatrix}2+i & 1 & 3\\ 1 & 2 & 3+2i\\ 3 & 3+2i & 4\end{bmatrix}$.

09. Skew-symmetric matrix: A square matrix $A=\left[a_{ij}\right]$ is said to be a *skew-symmetric matrix* if $A^T=-A$ *i.e.*, if $A=\left[a_{ij}\right]$ then, $A^T=\left[a_{ji}\right]=-\left[a_{ij}\right]\Rightarrow A^T=-A$.

For example: $\begin{bmatrix}0 & 1 & -3\\ -1 & 0 & 5\\ 3 & -5 & 0\end{bmatrix},\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$.

❖ ***Facts you should know:***

- Note that $\left[a_{ji}\right]=-\left[a_{ij}\right]\Rightarrow\left[a_{ii}\right]=-\left[a_{ii}\right]\Rightarrow 2\left[a_{ii}\right]=0$ (Replacing j by i)
  That is, ***all the diagonal elements in a skew-symmetric matrix are zero***.
- The matrices AA^T and A^TA are symmetric matrices.
- For any square matrix A, the matrix A + A^T is a symmetric matrix and A – A^T is a skew-symmetric matrix *always*.
- Also note that ***any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix*** i.e., $A=\frac{1}{2}(P)+\frac{1}{2}(Q)$, where P = A + A^T is a symmetric matrix and Q = A – A^T is a skew-symmetric matrix.

10. Orthogonal matrix: A matrix A is said to be orthogonal if $A.A^T$ = I where A^T is transpose of A.

11. Invertible Matrix: If A is a square matrix of order *m* and if there exists another square matrix B of the same order *m*, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}. A matrix having an inverse is said to be ***invertible***.

It is to note that if B is inverse of A, then A is also the inverse of B. In other words, if it is known that $AB=BA=I$ then, $A^{-1}=B\Leftrightarrow B^{-1}=A$.

12. Elementary Operations or Transformations of a Matrix: The following three operations applied on the row (or column) of a matrix are called elementary row (or column) transformations-

a) *Interchange of any two rows (or columns)**:*** When i^{th} row (or column) of a matrix is interchanged with the j^{th} row (or column), it is denoted as $R_i\leftrightarrow R_j$ (or $C_i\leftrightarrow C_j$).

b) Multiplying all elements of a row (or column) of a matrix by a non-zero scalar: When the i^{th} row (or column) of a matrix is multiplied by a scalar *k*, it is denoted as $R_i \to kR_i$ (or $C_i \to kC_i$).

c) Adding to the elements of a row (or column), the corresponding elements of any other row (or column) multiplied by any scalar k: When *k* times the elements of j^{th} row (or column) is added to the corresponding elements of the i^{th} row (or column), it is denoted as $R_i \to R_i + kR_j$ (or $C_i \to C_i + kC_j$).

NOTE: In case, after applying one or more elementary row (or column) operations on A = IA (or A = AI), if we obtain **all zeros in one or more rows** of the matrix A on LHS, then $\mathbf{A^{-1}}$ **does not exist.**

13. Inverse or reciprocal of a square matrix: If A is a square matrix of order n, then a matrix B (if such a matrix exists) is called the inverse of A if $AB = BA = I_n$. Also note that the inverse of a square matrix A is denoted by A^{-1} and we write, $A^{-1} = B$.

- *Inverse of a square matrix A exists if and only if A is non-singular matrix i.e.,* $|A| \neq 0$ (*explained later in the Determinant section*).
- *If B is inverse of A, then A is also the inverse of B.*

14. Algorithm to find Inverse of a matrix by Elementary Operations or Transformations:

➲ **By Row Transformations:**

STEP1- Write the given square matrix as $A = I_nA$.

STEP2- Perform a sequence of elementary row operations successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result $I_n = BA$.

STEP3- Matrix B is the inverse of A. So, write $A^{-1} = B$.

➲ **By Column Transformations:**

STEP1- Write the given square matrix as $A = AI_n$.

STEP2- Perform a sequence of elementary column operations successively on A on the LHS and post-factor I_n on the RHS till we obtain the result $I_n = AB$.

STEP3- Matrix B is the inverse of A. So, write $A^{-1} = B$.

WORKED OUT ILLUSTRATIVE EXAMPLES

Ex01. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{[i+2j]^2}{2}$.

Sol. Consider $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be the required matrix.

As $a_{ij} = \frac{[i+2j]^2}{2}$, so we have $a_{11} = \frac{[1+2(1)]^2}{2} = \frac{9}{2}$, $a_{12} = \frac{25}{2}$, $a_{21} = 8$, $a_{22} = 18$.

So the required matrix is $A = \begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$.

Ex02. Find the value of a if $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$.

Sol. We have $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

By equality of matrices, we get : a – b = –1, 2a + c = 5, 2a – b = 0 and 3c + d = 13.
Solving these equations, we get : a = 1.

Ex03. Find the matrix *X*, if $X + Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$ and $X - Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Sol. We have $X+Y=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}$ and $X-Y=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$.

On adding these two, we get : $(X+Y)+(X-Y)=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}+\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$

$$\Rightarrow \quad 2X=\begin{pmatrix}10 & 0\\ 2 & 8\end{pmatrix} \quad \Rightarrow \quad X=\begin{pmatrix}5 & 0\\ 1 & 4\end{pmatrix}.$$

Ex04. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=O$.

Sol. We have $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$ $\therefore$ $A^2=A.A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\Rightarrow A^2=\begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}\ldots(i)$

$-5A=-5\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-15 & -5\\ 5 & -10\end{bmatrix}$ $\ldots(ii)$

And, $7\mathrm{I}=7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$ $\ldots(iii)$

Adding these three equations, we get : $A^2-5A+7I=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}+\begin{bmatrix}-15 & -5\\ 5 & -10\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$

$\Rightarrow A^2-5A+7I=\begin{bmatrix}8-15+7 & 5-5+0\\ -5+5+0 & 3-10+7\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ $\Rightarrow A^2-5A+7I=\mathrm{O}$. **[H.P.]**

Ex05. Express the matrix $A=\begin{bmatrix}3 & 2 & 3\\ 4 & 5 & 3\\ 2 & 4 & 5\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Sol. We have $A=\begin{bmatrix}3 & 2 & 3\\ 4 & 5 & 3\\ 2 & 4 & 5\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}3 & 4 & 2\\ 2 & 5 & 4\\ 3 & 3 & 5\end{bmatrix}$ $\therefore$ $A+A^T=\begin{bmatrix}6 & 6 & 5\\ 6 & 10 & 7\\ 5 & 7 & 10\end{bmatrix}$ and $A-A^T=\begin{bmatrix}0 & -2 & 1\\ 2 & 0 & -1\\ -1 & 1 & 0\end{bmatrix}$

Let $P=\frac{1}{2}\left(A+A^T\right)=\begin{bmatrix}3 & 3 & 5/2\\ 3 & 5 & 7/2\\ 5/2 & 7/2 & 5\end{bmatrix}$ and $Q=\frac{1}{2}\left(A-A^T\right)=\begin{bmatrix}0 & -1 & 1/2\\ 1 & 0 & -1/2\\ -1/2 & 1/2 & 0\end{bmatrix}$.

We observe that, $P^T=\begin{bmatrix}3 & 3 & 5/2\\ 3 & 5 & 7/2\\ 5/2 & 7/2 & 5\end{bmatrix}^T=\begin{bmatrix}3 & 3 & 5/2\\ 3 & 5 & 7/2\\ 5/2 & 7/2 & 5\end{bmatrix}=P$

and, $Q^T=\begin{bmatrix}0 & -1 & 1/2\\ 1 & 0 & -1/2\\ -1/2 & 1/2 & 0\end{bmatrix}^T=\begin{bmatrix}0 & 1 & -1/2\\ -1 & 0 & 1/2\\ 1/2 & -1/2 & 0\end{bmatrix}\Rightarrow Q^T=-\begin{bmatrix}0 & -1 & 1/2\\ 1 & 0 & -1/2\\ -1/2 & 1/2 & 0\end{bmatrix}=-Q$

Thus, it is clear that P is symmetric matrix and Q is skew-symmetric matrix.

Hence we have, $P+Q=\begin{bmatrix}3 & 3 & 5/2\\ 3 & 5 & 7/2\\ 5/2 & 7/2 & 5\end{bmatrix}+\begin{bmatrix}0 & -1 & 1/2\\ 1 & 0 & -1/2\\ -1/2 & 1/2 & 0\end{bmatrix}$ $\Rightarrow P+Q=\begin{bmatrix}3 & 2 & 3\\ 4 & 5 & 3\\ 2 & 4 & 5\end{bmatrix}=A$.

Thus, we've expressed matrix A as the sum of a symmetric matrix & a skew-symmetric matrix.

Ex06. If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, then show that $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}\forall n\in\mathbb{N}$.

Sol. We shall be using principle of mathematical induction to prove this.

Let P(n) : $\mathrm{A}^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}\forall n\in\mathrm{N}$

For n = 1, P(1) : $\mathrm{A}^1=\begin{bmatrix}1+2(1) & -4(1)\\ 1 & 1-2(1)\end{bmatrix}=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\mathrm{A}$ [Given $\mathrm{A}=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$

$\therefore$ P(1) is true.

Assume that P(k) is true for $k \in N$ i.e., P(k) : $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \forall k \in N$ …(i)

We have to show that P(k + 1) is also true whenever P(k) is true i.e.,

P(k + 1) : $A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{bmatrix}$

Consider LHS : $A^{k+1} = A^k A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ [By using (i)

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix} = \begin{bmatrix} 3-2k & -4-4k \\ k+1 & -1-2k \end{bmatrix} = \text{RHS.}$$

$\therefore$ P(k + 1) is also true.

Hence by principle of mathematical induction, P(n) is always true for all natural numbers n.

Ex07. Using elementary transformations, find the inverse of the matrix : $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$.

Sol. Let $A = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$

By using elementary row transformations, we have: $A = IA$

$\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$ [Applying $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 2R_1$

$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} A$ [Applying $R_2 \to \frac{1}{9} R_2$

$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -7/9 \\ 0 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/9 & 0 \\ -2 & 0 & 1 \end{pmatrix} A$ [Applying $R_1 \to R_1 - 3R_2$, $R_3 \to R_3 + 5R_2$

$\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & 0 & 1/9 \end{pmatrix} = \begin{pmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -1/3 & 5/9 & 1 \end{pmatrix} A$ [Applying $R_3 \to 9R_3$

$\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -3 & 5 & 9 \end{pmatrix} A$ [Applying $R_1 \to R_1 - \frac{1}{3} R_3$, $R_2 \to R_2 + \frac{7}{9} R_3$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix} A$ $\therefore I = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix} A$

$\because I = A^{-1}A$ So, $A^{-1} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}$.

Hence, the inverse of matrix $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}$.

EXERCISE FOR PRACTICE

(BASED ON ALGEBRA OF MATRICES)

VERY SHORT ANSWER TYPE QUESTIONS

Q01. If $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$, find the matrix AB.

Q02. **(a)** What is the element a_{23} in the matrix $A = \lambda\left[a_{ij}\right]_{3\times 3}$ s.t. $a_{ij} = \frac{2(9i-j)}{3}$?

(b) For a 2×2 matrix $A = \left[a_{ij}\right]$, whose elements are given by $a_{ij} = \frac{i}{j}$, write the value of a_{12}.

(c) Write the element a_{23} of a 3×3 matrix $\mathrm{A} = (\mathrm{a}_{\mathrm{ij}})$ whose elements a_{ij} are given by $\mathrm{a}_{\mathrm{ij}} = \frac{|\mathrm{i}-\mathrm{j}|}{2}$.

Q03. **(a)** How many matrices of order 2×3 are possible with each entry 0 or 1?

(b) What is the number of all possible matrices of order 3×3 with each entry as 0 or 1?

Q04. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 3 \end{bmatrix}$. For what value of *x*, *A* will be a scalar matrix?

Q05. If a matrix has 12 elements, what are the possible orders it can have?

Q06. A matrix *X* has *a*+*b* rows and $a+2$ columns while the matrix *Y* has *b*+1 rows and *a*+3 columns. Both the matrices *XY* and *YX* exist. Find the values of *a* and *b*.

Q07. **(a)** If *A* is a matrix of 2×3 and *B* is of 3×5, what is the order of $(AB)^T$?

(b) If *A* is 3×4 matrix and *B* is a matrix such that $A^T B$ and BA^T are both defined. Then what is the order of matrix *B*?

Q08. If it is given that $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ then, find A^2.

Q09. **(a)** Construct a matrix $\left[a_{ij}\right]_{4\times 3}$ such that $a_{ij} = \frac{i-j}{i+j}$.

(b) Construct a 3×2 matrix *B* such that $b_{ij} = \frac{|i-2j|}{3}$.

(c) Construct a 2×3 matrix *A* whose elements are given by $a_{ij} = \begin{cases} i-2j, & if \quad i>j \\ i-j, & if \quad i=j \\ -i+3j, & if \quad i<j \end{cases}$.

Q10. **(a)** Solve the matrix equation: $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ 5 \end{bmatrix} = \mathrm{O}$.

(b) Find the values of *x* from the matrix equation: $\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 7x+y \\ 2y & 10 \end{pmatrix}$.

(c) Solve the following matrix equation for x : $[\mathrm{x} \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = \mathrm{O}$.

(d) For what values of x : $[1 \quad 2 \quad 1]\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

Q11. **(a)**Find matrix A and B if $2A - B = \begin{bmatrix} 4 & -6 \\ -4 & 2 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$.

(b) Find matrix X and Y if $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$.

Q12. If $A = diag\begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$ and $B = diag\begin{pmatrix} 2 & 3 & -1 \end{pmatrix}$, find $3A + 4B$.

Q13. If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix *X*, such that $2A + 3X = 5B$.

Q14. If $A = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$, then for what value of ω is A an identity matrix?

Q15. If $A = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ and $A + A^T = \mathrm{I}_2$ then, what is the value of ϑ?

Q16. If $A = \begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$ then, verify that $A'A = \mathrm{I}$.

Q17. Simplify: $\cos\omega\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} + \sin\omega\begin{bmatrix} \sin\omega & -\cos\omega \\ \cos\omega & \sin\omega \end{bmatrix}$.

Q18. Evaluate: $1 - \omega^2 - \kappa\eta$ if $A = \begin{bmatrix} \omega & \kappa \\ \eta & -\omega \end{bmatrix}$ satisfies the equation $A^2 = \mathrm{I}$.

Q19. If A is a square matrix such that $A^2 = A$ then, what is the value of $(\mathrm{I} + A)^3 - 7A$?

Q20. Show that the elements on the main diagonal of a skew-symmetric matrix are all zero.

Q21. **(a)** Find the value of a and b if $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$.

(b) If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A.

(c) If $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{pmatrix}$, write the value of a – 2b.

Q22. Solve for the *unknown variables* viz. *w, x, y, z, a, b, c* (as the case may be) in the followings:

a) $\begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix} = 2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

b) $\begin{bmatrix} x+y & 3 \\ 7 & xy \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 7 & -12 \end{bmatrix}$

c) $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - \begin{bmatrix} 5x \\ 6y \end{bmatrix} = 3\begin{bmatrix} -2 \\ -3 \end{bmatrix}$

d) $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$

e) $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+a \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

f) $\begin{bmatrix} 2x+y & 3y \\ 0 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$

g) $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

h) $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}$ such that $A^2 + B^2 = (A+B)^2$

Q23. Find the values of x for which the matrix product $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ equals an identity matrix.

Q24. Find the values of x, y, z if the matrix $A=\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies the equation $A^T A = \text{I}$.

Q25. If A and B are two matrices such that $A+B$ and AB are both defined, then:
(a) A and B can be any matrices
(b) A and B are square matrices of same order
(c) A and B are square matrices not necessarily of same order
(d) None of these

Q26. Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew- symmetric matrix.

Q27. If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute. [A and B commute means $AB = BA$.]

Q28. If A and B are symmetric matrices, prove that $AB - BA$ is a skew-symmetric matrix.

Q29. Show that the matrix $B^T AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Q30. If B is skew-symmetric matrix, write whether ABA' is symmetric or skew-symmetric.

Q31. Write the values of p and q such that the matrix A is skew symmetric, $A=\begin{pmatrix} 0 & 5 & -3 \\ -5 & p & 4 \\ q & -4 & 0 \end{pmatrix}$.

Q32. If $Z=\begin{Vmatrix} 10 & -2 \\ -5 & 1 \end{Vmatrix}$ then find Z^{-1}, if it exists. Use elementary operations.

Q33. If $A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then, write the matrix A^4.

Q34. If $A=\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}$, then find the value of A^{20}.

Q35. If $2\begin{pmatrix} 3 & 4 \\ 5 & \text{x} \end{pmatrix}+\begin{pmatrix} 1 & \text{y} \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 7 & 0 \\ 10 & 5 \end{pmatrix}$, then find (x – y).

Q36. Use elementary column operations $\text{C}_2 \to \text{C}_2 - 2\text{C}_1$ in the matrix equation
$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

SHORT & LONG ANSWER TYPE QUESTIONS

Q01. If l_i, m_i, n_i; $i = 1, 2, 3$ denote the direction cosines of three mutually perpendicular lines in the space, then prove that $A^T A = \text{I}$ such that $A=\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$. [Based on the concept from Three Dimensional Geometry, NCERT Chapter 11.]

Q02. If $A=\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$, $C=\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$ and $AB - CD = \text{O}$ then, find the matrix D.

Q03. If $A=\begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ and $C=\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, then verify that $(A+B)C = AC + BC$.

☞ This property is known as the ***distributive property of matrix addition***.

Q04. Find the matrix A, if:

a) $\begin{bmatrix} -2 & 4 & -2 \\ 3 & 7 & 3 \end{bmatrix} + A = \begin{bmatrix} -1 & 2 & 6 \\ 4 & 5 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix} - A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$

c) $A\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ **d)** $A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

e) $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = 6\,\mathrm{I}_2$ **f)** $\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} A = \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$

Q05. Find the matrix A, satisfying the matrix equation: $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \mathrm{I}$.

Q06. If it is known that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, find A.

Q07. Define a symmetric and skew-symmetric matrix. Prove that for the matrix $X = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$, $X - X^T$ is skew- symmetric matrix whereas $X + X^T$, XX^T and $X^T X$ is symmetric matrix.

Q08. Express the matrix $\begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix}$ as the sum of symmetric and skew-symmetric matrix.

Q09. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$ where $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

Q10. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, show that $AB - BA$ is a skew-symmetric matrix.

Q11. Express $\begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Q12. Express $A = \begin{bmatrix} -3 & 6 & 0 \\ 4 & -5 & 8 \\ 0 & -7 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Q13. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and $f(x) = x^2 - 5x - 6$ then, find $f(A)$.

Q14. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ and $f(x) = x^2 - 5x + 6$ then, find $f(A)$.

Q15. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ then, show that A is a root of the cubic equation $x^3 - 6x^2 + 7x + 2 = 0$.

Q16. **a)** If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the value of k so that $A^2 = kA - 2\mathrm{I}$.

b) If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $A^{-1} = k\,A$, find the value of k.

c) If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19}A$.

Q17. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$. Show that $f(A) = \mathrm{O}$. Use this result to find A^5.

Q18. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and $A^2 - 2B + 7\mathrm{I} = \mathrm{O}$ then, find the matrix B.

Q19. If $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ then, prove that $\mathrm{A}^3 - 23\mathrm{A} - 40\mathrm{I} = \mathrm{O}$.

Q20. If $A=\begin{bmatrix}3&2&0\\1&4&0\\0&0&5\end{bmatrix}$ then, prove that $(A-5\mathrm{I})(A-2\mathrm{I})=\mathrm{O}$. Hence find A^{-1}.

Q21. For the matrix $A=\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}$, verify that $A^3-6A^2+5A+11\,\mathrm{I}=\mathrm{O}$. Hence find A^{-1}.

Q22. Show that $\begin{bmatrix}5&3\\-1&-2\end{bmatrix}$ satisfies the equation $x^2-3x-7=0$. Thus find the inverse of given matrix.

Q23. If $A=\begin{bmatrix}3&1\\7&5\end{bmatrix}$ then, find x and y so that $A^2+x\mathrm{I}-yA=\mathrm{O}$. Hence find A^{-1}.

Q24. If $A=\begin{bmatrix}3&2\\1&1\end{bmatrix}$ then, find a and b so that $A^2+aA+b\mathrm{I}=\mathrm{O}$. Hence find A^{-1}.

Q25. If $A=\begin{bmatrix}-1&2\\3&0\end{bmatrix}$ and $B=\begin{bmatrix}1&-1\\2&2\end{bmatrix}$ then, verify that $A^{-1}B^{-1}=(BA)^{-1}$.

Q26. If $A=\begin{bmatrix}2&1\\3&4\end{bmatrix}$ and $B^{-1}=\begin{bmatrix}-1&0\\3&4\end{bmatrix}$ then, find $(AB)^{-1}$.

Q27. If $A=\begin{bmatrix}2&-1&2\\-4&2&1\end{bmatrix}$ and $B=\begin{bmatrix}0&-1\\1&3\\4&5\end{bmatrix}$ then, verify that $(AB)^T=B^TA^T$.

Q28. Prove the followings by the **Principle *Of* Mathematical Induction** :

a) $A^n=\begin{bmatrix}\cos n\theta&\sin n\theta\\-\sin n\theta&\cos n\theta\end{bmatrix}$, $n\in\mathrm{N}$ if $A=\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$

b) $A^n=\begin{bmatrix}\cos n\theta&i\sin n\theta\\i\sin n\theta&\cos n\theta\end{bmatrix}$ for all $n\in\mathrm{N}$ if $A=\begin{bmatrix}\cos\theta&i\sin\theta\\i\sin\theta&\cos\theta\end{bmatrix}$

c) $(a\mathrm{I}+bA)^n=a^n\mathrm{I}+na^{n-1}bA$, where I is the identity matrix of order 2 and $n\in\mathrm{N}$, if it is given that $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$.

d) $A^n=\begin{bmatrix}1+2n&-4n\\n&1-2n\end{bmatrix}, n\in Z^+$ if $A=\begin{bmatrix}3&-4\\1&-1\end{bmatrix}$

e) $A^n=\begin{bmatrix}3^{n-1}&3^{n-1}&3^{n-1}\\3^{n-1}&3^{n-1}&3^{n-1}\\3^{n-1}&3^{n-1}&3^{n-1}\end{bmatrix}, n\in\mathrm{N}$ if $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

f) $A^n=\begin{bmatrix}a^n&\dfrac{b(a^n-1)}{a-1}\\0&1\end{bmatrix}, n\geq 0$ if $A=\begin{bmatrix}a&b\\0&1\end{bmatrix}$ where $a\neq 1$

Q29. **(a)** If $A=diag\begin{pmatrix}a&b&c\end{pmatrix}$, show that $A^n=diag\begin{pmatrix}a^n&b^n&c^n\end{pmatrix}$ for all positive integer n.

(b) If A and B are square matrices of the same order such that AB = BA, then prove by induction that $\mathrm{AB}^n=\mathrm{B}^n\mathrm{A}$. Further, prove that $(\mathrm{AB})^n=\mathrm{A}^n\mathrm{B}^n$ for all $n\in\mathrm{N}$.

Q30. If $A=\begin{bmatrix}0&-\tan\dfrac{x}{2}\\\tan\dfrac{x}{2}&0\end{bmatrix}$ and I is an identity matrix then, show that $(\mathrm{I}+A)=(\mathrm{I}-A)\begin{bmatrix}\cos x&-\sin x\\\sin x&\cos x\end{bmatrix}$.

Q31. If $\varphi(x)=\begin{bmatrix}\cos x&-\sin x&0\\\sin x&\cos x&0\\0&0&1\end{bmatrix}$ then, show that $\varphi(x)\varphi(y)=\varphi(x+y)$.

Q32. If $A=\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}$ then, verify that $A(\text{adj}A)=|A|\,\text{I}$. Also find A^{-1}.

Q33. By using **elementary operations (transformations)**, find the inverse of matrix A (if it exists) in the followings :

a) $\begin{bmatrix}1&2\\2&-1\end{bmatrix}$ **b)** $\begin{bmatrix}3&10\\2&7\end{bmatrix}$ **c)** $\begin{bmatrix}1&-1\\2&3\end{bmatrix}$ **d)** $\begin{bmatrix}6&-3\\-2&1\end{bmatrix}$

e) $\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$ **f)** $\begin{bmatrix}1&-1&2\\0&2&-3\\3&-2&4\end{bmatrix}$ **g)** $\begin{bmatrix}2&-3&3\\2&2&3\\3&-2&2\end{bmatrix}$ **h)** $\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$

Q34. Prove that the product of matrices $\begin{bmatrix}\cos^2\theta&\cos\theta\sin\theta\\\cos\theta\sin\theta&\sin^2\theta\end{bmatrix}$ and $\begin{bmatrix}\cos^2\beta&\cos\beta\sin\beta\\\cos\beta\sin\beta&\sin^2\beta\end{bmatrix}$ is a null matrix when θ and β differ by an odd multiple of $\frac{\pi}{2}$.

Q35. Show that: $\begin{bmatrix}1&-\tan\frac{\theta}{2}\\\tan\frac{\theta}{2}&1\end{bmatrix}\begin{bmatrix}1&\tan\frac{\theta}{2}\\-\tan\frac{\theta}{2}&1\end{bmatrix}^{-1}=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$.

Q36. Using $1+\omega+\omega^2=0$ and $\omega^3=1$, show the following:

$$\left(\begin{bmatrix}1&\omega&\omega^2\\\omega&\omega^2&1\\\omega^2&1&\omega\end{bmatrix}+\begin{bmatrix}\omega&\omega^2&1\\\omega^2&1&\omega\\\omega&\omega^2&1\end{bmatrix}\right)\begin{bmatrix}1\\\omega\\\omega^2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

☞ The identities $1+\omega+\omega^2=0$ and $\omega^3=1$ are known as ***complex cube root of unity***.

Q37. If $A=\begin{bmatrix}1&2&2\\2&1&-2\\a&2&b\end{bmatrix}$ is a matrix satisfying $AA^T=9\text{I}_3$, then find the values of *a* and *b*.

Q38. Let $A=\begin{bmatrix}1&\sin\alpha&1\\-\sin\alpha&1&\sin\alpha\\-1&-\sin\alpha&1\end{bmatrix}$, where $0\le\alpha\le2\pi$. Then which of the following is true:

a) $\det(A)=0$ b) $\det(A)\in[2,4]$ c) $\det(A)\in(2,4)$ d) $\det(A)\in(2,\infty)$

OR Evaluate the determinant $\Delta=\begin{vmatrix}1&\sin\theta&1\\-\sin\theta&1&\sin\theta\\-1&-\sin\theta&1\end{vmatrix}$. Also prove that $2\le\Delta\le4$.

Q39. In XII class examination, 25 students from school A and 35 students from school B appeared. Only 20 students from each school could get through the examination. Out of them, 15 students from school A and 10 students from school B secured full marks. Write down this information in matrix from.

Q40. Let $\text{A}=\begin{bmatrix}8&16\\32&48\end{bmatrix}$, where first row represents the number of table fans and second row represents the number of ceiling fans which two manufacturing units x and y makes in one day. Compute 7A and, state what does it represents?

Q41. If $\text{A}=\begin{pmatrix}2&0&1\\2&1&3\\1&-1&0\end{pmatrix}$ find $\text{A}^2-5\text{A}+4\text{I}$ and hence find a vector X such that $\text{A}^2-5\text{A}+4\text{I}+\text{X}=\text{O}$.

Q42. If $\text{A}=\begin{bmatrix}1&-2&3\\0&-1&4\\-2&2&1\end{bmatrix}$, find $(\text{A}')^{-1}$.

Q43. Three schools A, B and C organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of ₹25, ₹100 and ₹50 each. The number of articles sold are given below :

Article \ School	A	B	C
Hand fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also find the total funds collected for the purpose. Write one value generated by the above situation.

DETERMINANTS, ITS PROPERTIES & APPLICATIONS

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Determinants, Minors & Cofactors:

a) Determinant: A unique number (real or complex) can be associated to every square matrix $A=\left[a_{ij}\right]$ of order *m*. This number is called the determinant of the square matrix A, where $a_{ij}=(i,\ j)^{th}$ element of A.

For instance, if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, determinant of matrix A is written as $|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=\det.(A)$ and its value is given by $ad-bc$.

b) Minors: Minors of an element a_{ij} of a determinant *(or a determinant corresponding to matrix A)* is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence we can get 9 minors corresponding to the 9 elements of a third order $(i.e., 3\times 3)$ determinant.

c) Cofactors: Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by, $A_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.

02. Adjoint of a square matrix: Let $A=\left[a_{ij}\right]$ be a square matrix. Also assume $B=\left[A_{ij}\right]$ where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the ***adjoint of matrix A*** and it is denoted by "***adj.*A**".

> ***To find adjoint of a 2×2 matrix :*** Follow this, $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow adj.A=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example, consider a square matrix of order 3 as $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$ then, in order to find the adjoint of matrix A, we find a matrix B (formed by the cofactors of elements of matrix A as mentioned above in the definition) *i.e.,* $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj.A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$.

03. Singular matrix & Non-singular matrix:

a) Singular matrix: A square matrix A is said to be *singular if* $|A| = 0$ *i.e.*, its **determinant is *zero***.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}$.

b) Non-singular matrix: A square matrix A is said to be *non-singular if* $|A| \neq 0$.

e.g. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$.

❖ *A square matrix A is* ***invertible*** *if and only if A is* ***non-singular***.

04. Algorithm to find A^{-1} by Determinant method:

STEP1- Find |A|.

STEP2- If |A| = 0 then, write **"A is a singular matrix and hence not invertible"**. Else write **"A is a non-singular matrix and hence invertible"**.

STEP3- Calculate the cofactors of elements of matrix A.

STEP4- Write the matrix of cofactors of elements of A and then obtain its transpose to get *adj.*A (*i.e.*, *adjoint* A).

STEP5- Find the inverse of A by using the relation $A^{-1} = \frac{1}{|A|} adj.A$.

05. Properties associated with various operations of Matrices & the Determinants:

a) AB = I = BA

b) $AA^{-1} = I$ or $A^{-1}I = A^{-1}$

c) $(AB)^{-1} = B^{-1}A^{-1}$

d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

e) $(A^{-1})^{-1} = A$

f) $(A^T)^{-1} = (A^{-1})^T$

g) $A(adj.A) = (adj.A)A = |A|\, I$

h) $adj.(AB) = (adj.B)(adj.A)$

i) $adj.(A^T) = (adj.A)^T$

j) $(adj.A)^{-1} = (adj.A^{-1})$

k) $|adj.A| = |A|^{n-1}$, *if* $|A| \neq 0$, where *n* is order of A

l) $|AB| = |A||B|$

m) $|A.\, adj.A| = |A|^n$, where *n* is order of A

n) $|A^{-1}| = \frac{1}{|A|}$, iff matrix A is invertible

o) $|A| = |A^T|$

- $|kA| = k^n |A|$ where *n* is order of square matrix *A* and *k* is any scalar.
- If A is a non-singular matrix (*i.e.*, when $|A| \neq 0$) of order *n*, then $|adj.A| = |A|^{n-1}$.
- If A is a non-singular matrix of order *n*, then $adj.(adj.A) = |A|^{n-2} A$.

06. Properties of Determinants:

a) If any two rows or columns of a determinant are *proportional* or *identical*, then its value is equal to *zero*.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$ [As R_1 and R_3 are the same.

b) The value of a determinant remains *unchanged* if its rows and columns are *interchanged*.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Here rows and columns have been interchanged, but there is *no effect* on the value of determinant.

c) If each element of a row or a column of a determinant is multiplied by a constant *k*, then the value of new determinant is *k* times the value of the original determinant.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta_1 = k\Delta$.

d) If *any two* rows or columns are *interchanged*, then the determinant retains its *absolute* value, but its *sign* is changed.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \Rightarrow \Delta_1 = -\Delta$ [Here $R_1 \leftrightarrow R_3$.

e) If every element of some column or row is the *sum* of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as given in the original determinant.

e.g. $\Delta = \begin{vmatrix} a_1+\alpha & b_1 & c_1 \\ a_2+\beta & b_2 & c_2 \\ a_3+\gamma & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b_3 & c_3 \end{vmatrix}$.

07. Area of triangle: Area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ Sq. units.} \quad \ldots(A)$$

➲ Since area is a positive quantity, we take **absolute value of the determinant** in (A).

➲ If the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are **collinear** then $\Delta = 0$.

➲ The **equation of a line** passing through the points (x_1, y_1) and (x_2, y_2) can be obtained by the expression given here : $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.

08. Solutions of System of Linear equations:

a) Consistent and Inconsistent system: A system of equations is consistent if it has *one or more* solutions otherwise it is said to be an inconsistent system. In other words an inconsistent system of equations has *no solution*.

b) Homogeneous and Non-homogeneous system: A system of equations $AX = B$ is said to be a homogeneous system if $B = 0$. Otherwise it is called a non-homogeneous system of equations.

09. Solving of system of equations by Matrix method [Inverse Matrix Method]:

Consider the following system of equations,

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$

$a_3x + b_3y + c_3z = d_3$.

STEP1- Assume $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

STEP2- Find |A|. Now there may be following situations:

a) $|A| \neq 0 \Rightarrow A^{-1}$ exists. It implies that the given system of equations is *consistent* and therefore, the system has ***unique solution***. In that case, write

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B \qquad \left[\text{where } A^{-1} = \frac{1}{|A|}(adj.A)\right.$$

$\Rightarrow$ Then by using the ***definition of equality of matrices***, we can get the values of *x, y* and *z*.

b) $|A| = 0 \Rightarrow A^{-1}$ does not exist. It implies that the given system of equations may be *consistent* or *inconsistent*. In order to check proceed as follow:

$\Rightarrow$ Find (*adj*.A)B. Now we may have either $(adj.A)B \neq 0$ or $(adj.A)B = 0$.

- If $(adj.A)B = 0$, then the given system may be *consistent* or *inconsistent*. To check, put $z = k$ in the given equations and proceed in the same manner in the new *two variables* system of equations assuming $d_i - c_ik$, $1 \le i \le 3$ as constant.
- And if $(adj.A)B \neq 0$, then the given system is *inconsistent* with *no solutions*.

WORKED OUT ILLUSTRATIVE EXAMPLES

Ex01. If the points $(2,-3),(k,-1)$ and $(0,4)$ are collinear, find the value of *k*.

Sol. The points $(2,-3),(k,-1)$ and $(0,4)$ are collinear. So we must have,

$$\begin{vmatrix} 2 & -3 & 1 \\ k & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = 0 \qquad (\text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$$

$$\Rightarrow \quad = \begin{vmatrix} 2 & -3 & 1 \\ k-2 & 2 & 0 \\ -2 & 7 & 0 \end{vmatrix} = 0 \qquad (\text{On expanding along } C_3)$$

$$\Rightarrow \quad 1\begin{vmatrix} k-2 & 2 \\ -2 & 7 \end{vmatrix} = 0 \quad \Rightarrow \quad 7(k-2) + 4 = 0 \quad \Rightarrow k = \frac{10}{7}.$$

Ex02. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of *x*.

Sol. We have $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix} \Rightarrow (x+1)(x+2) - (x-1)(x-3) = 4 \times 3 - (-1) \times 1 \quad \therefore x = 2$.

Ex03. Using properties of determinant, prove that :

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

Sol. **LHS :** Let $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$ $\qquad$ (Applying $R_3 \to R_3 + R_1$)

$$=\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$ (Taking $a+b+c$ common from R_3)

$$=(a+b+c)\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix}$$ (Applying $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$)

$$=(a+b+c)\begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ 0 & 0 & 1 \end{vmatrix}$$ (Taking $(a-b)$ & $(b-c)$ common from C_1 & C_2 respectively)

$$=(a-b)(b-c)(a+b+c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ 0 & 0 & 1 \end{vmatrix}$$ (Applying $C_2 \to C_2 - C_1$)

$$=(a-b)(b-c)(a+b+c)\begin{vmatrix} 1 & 0 & c \\ a+b & c-a & c^2 \\ 0 & 0 & 1 \end{vmatrix}$$ (Taking $(c-a)$ common from C_2)

$$=(a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix} 1 & 0 & c \\ a+b & 1 & c^2 \\ 0 & 0 & 1 \end{vmatrix}$$ (Expanding along R_3)

$$=(a-b)(b-c)(c-a)(a+b+c)\left(1\begin{vmatrix} 1 & 0 \\ a+b & 1 \end{vmatrix}\right)$$

$$=(a-b)(b-c)(c-a)(a+b+c)(1(1-0))$$

$$=(a-b)(b-c)(c-a)(a+b+c)=\textbf{RHS.}$$ **[H.P.]**

Ex04. Prove that : $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$**.**

Sol. **LHS :** Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

(Taking a, b and c common from R_1, R_2, R_3 respectively)

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ (Taking a,b,c common from C_1, C_2, C_3 respectively)

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ (Applying $C_1 \to C_1 + C_2, C_2 \to C_2 + C_3$)

$$= a^2b^2c^2\begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$ (Expanding along R_2)

$$= a^2b^2c^2\left(-1\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}\right)$$

$$\Rightarrow \Delta = a^2b^2c^2(-1(0-4)) = 4a^2b^2c^2 = \textbf{RHS.}$$ **[H.P.]**

Ex05. Solve the following system of equations using matrix method :
$\boldsymbol{x+2y+z=7,\ x+3z=11,\ 2x-3y=1}$**.**

Sol. The given system of equations is :

$x+2y+z=7,$
$x+3z=11,$
$2x-3y=1$

Let $A=\begin{bmatrix}1&2&1\\1&0&3\\2&-3&0\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}7\\11\\1\end{bmatrix}.$

Now, $|A|=\begin{vmatrix}1&2&1\\1&0&3\\2&-3&0\end{vmatrix}=1(0+9)-2(0-6)+1(-3-0)=18\neq 0$. So, A^{-1} exists.

Let A_{ij} be the cofactors of elements a_{ij} in $A=\left[a_{ij}\right]$. Then, we have :

$A_{11}=9,\quad A_{21}=-3,\quad A_{31}=6$
$A_{12}=6,\quad A_{22}=-2,\quad A_{32}=-2$
$A_{13}=-3,\quad A_{23}=7,\quad A_{33}=-2$

$\therefore\ adjA=\begin{bmatrix}9&-3&6\\6&-2&-2\\-3&7&-2\end{bmatrix}$ $\quad\therefore\ A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{18}\begin{bmatrix}9&-3&6\\6&-2&-2\\-3&7&-2\end{bmatrix}$

Now as $AX=B \Rightarrow A^{-1}AX=A^{-1}B$ [Pre-multiplying by A^{-1} both the sides

$\Rightarrow IX=A^{-1}B \Rightarrow X=A^{-1}B$

So, $X=\frac{1}{18}\begin{bmatrix}9&-3&6\\6&-2&-2\\-3&7&-2\end{bmatrix}\begin{bmatrix}7\\11\\1\end{bmatrix}$ *i.e.*, $X=\frac{1}{18}\begin{bmatrix}63-33+6\\42-22-2\\-21+77-2\end{bmatrix}=\frac{1}{18}\begin{bmatrix}36\\18\\54\end{bmatrix}=\begin{bmatrix}2\\1\\3\end{bmatrix}$

$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\3\end{bmatrix}$

Hence by *equality of matrices*, we get : $x=2,\ y=1,\ z=3$.

Ex06. A school wants to award its students for the values Honesty, Regularity and Hard-work with a total cash award of ₹6000. Three times the award money for Hard-work added to that given for Honesty amounts to ₹11000. The award money given for Honesty and Hard-work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hard-work, suggest one more value which the school must include for the wards.

Sol. Let the award money for the values of Honesty, Regularity and Hard-work be x, y and z (in ₹) respectively.

According to question, $x+y+z=6000,\ x+3z=11000,\ x-2y+z=0$.

Let $A=\begin{bmatrix}1&1&1\\1&0&3\\1&-2&1\end{bmatrix}, B=\begin{bmatrix}6000\\11000\\0\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Now, $|A|=\begin{vmatrix}1&1&1\\1&0&3\\1&-2&1\end{vmatrix}=6\neq 0 \Rightarrow A^{-1}$ exists.

Consider C_{ij} be the cofactors of element a_{ij} in matrix A, we have

$C_{11}=6,\quad C_{12}=2,\quad C_{13}=-2$
$C_{21}=-3,\quad C_{22}=0,\quad C_{23}=3$
$C_{31}=3,\quad C_{32}=-2,\quad C_{33}=-1$

So, $adjA=\begin{bmatrix}6 & -3 & 3\\ 2 & 0 & -2\\ -2 & 3 & -1\end{bmatrix}$. $\quad\therefore\ A^{-1}=\dfrac{1}{|A|}(adjA)=\dfrac{1}{6}\begin{bmatrix}6 & -3 & 3\\ 2 & 0 & -2\\ -2 & 3 & -1\end{bmatrix}$.

As, $X=A^{-1}B \quad \Rightarrow X=\dfrac{1}{6}\begin{bmatrix}6 & -3 & 3\\ 2 & 0 & -2\\ -2 & 3 & -1\end{bmatrix}\begin{bmatrix}6000\\ 11000\\ 0\end{bmatrix}=\dfrac{1}{6}\begin{bmatrix}3000\\ 12000\\ 21000\end{bmatrix} \quad \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}500\\ 2000\\ 3500\end{bmatrix}$

By equality of matrices, we get: $x=500,\ y=2000,\ z=3500$.

Hence, award money given for the value of Honesty = ₹500, award money given for the value of Regularity = ₹2000 and, award money given for the value of Hard-work = ₹3500.

The school must include the value of **Obedience** for the awards.

EXERCISE FOR PRACTICE

(BASED ON THE DETERMINANTS)

VERY SHORT ANSWER TYPE QUESTIONS

Q01. **(a)** Determine the value of the determinant: $\begin{vmatrix}a+ib & c+id\\ -c+id & a-ib\end{vmatrix}$.

(b) If ω is a complex cube root of unity, then find the value of : $\begin{vmatrix}1 & \omega & \omega^2\\ \omega & \omega^2 & 1\\ \omega^2 & 1 & \omega\end{vmatrix}$.

(c) Write the value of $\begin{vmatrix}\sin 20^{o} & \cos 20^{o}\\ -\sin 70^{o} & \cos 70^{o}\end{vmatrix}$.

Q02. Find the value of xy, if $\begin{vmatrix}3x^3 & 8\\ -4 & 4y^3\end{vmatrix}=-4$.

Q03. **(a)** If $\begin{vmatrix}3x & 1\\ 5 & -x\end{vmatrix}=\begin{vmatrix}-1 & 1\\ 5 & 2\end{vmatrix}$, find the value(s) of x.

(b) If $\begin{vmatrix}x+1 & x-1\\ x-3 & x+2\end{vmatrix}=\begin{vmatrix}4 & -1\\ 1 & 3\end{vmatrix}$, then write the value of x.

Q04. If $x\in R$, $0\le x\le\dfrac{\pi}{2}$, and $\begin{vmatrix}2\sin x & -1\\ 1 & \sin x\end{vmatrix}=\begin{vmatrix}3 & 0\\ -4 & \sin x\end{vmatrix}$, then find the values of x.

Q05. If $A=\left[a_{ij}\right]$ is a 3×3 matrix and A_{ij} denotes the co-factors of the corresponding elements a_{ij}'s then, what is the value of $a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}$?

Q06. If $A=\left[a_{ij}\right]$ is a 3×3 matrix and M_{ij}'s denotes the minors of the corresponding elements a_{ij}'s then, write the expression for the value of $|A|$ by expanding $|A|$ by third column.

Q07. **(a)** It is known that $A=2B$, where A and B are square matrices of third order and $|B|=5$. What is the value of $|A|$?

(b) If A is a square matrix such that $A(adjA)=5I$ then determine the value of $|A|$.

(c) If A is a square matrix of order 3 such that $|A|=5$ then determine the value of $|adjA|$.

(d) If A is a square matrix of order 3 such that $|adjA|=64$ then find $|A|$.

(e) If A is a non-singular square matrix such that $|A|=10$ then determine the value of $|A^{-1}|$.

(f) If A is a square matrix of order 3 such that $A(adjA)=5I$, find $|adjA|$.

Q08. **(a)** For what value of x, the matrix $\begin{bmatrix}5-x & x+1\\ 2 & 4\end{bmatrix}$ is singular?

(b) For what value of x, the matrix $\begin{pmatrix} 2-x & 3 \\ -5 & 1 \end{pmatrix}$ is non- invertible?

(c) For what value of x, the matrix $\begin{bmatrix} 7-2x & x+5 \\ 3 & 7 \end{bmatrix}$ is singular?

Q09. If k is a scalar and A is a square matrix of order n, then $|kA|$:

(a) $k|A|^n$ (b) $k|A|$ (c) $k^n|A|^n$ (d) $k^n|A|$

Q10. If A is invertible then, $\det\left(A^{-1}\right)$ is equal to:

(a) det(A) (b) 1 (c) $\dfrac{1}{\det(A)}$ (d) 0

Q11. If x, y, z are all non-zero real numbers then, $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}^{-1}$ is:

(a) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (b) $\dfrac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (c) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (d) $\dfrac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q12. Find the minor of the element 8 in the following determinant : $\Delta = \begin{vmatrix} 2 & 4 & 7 \\ 3 & 6 & 8 \\ -2 & -3 & 1 \end{vmatrix}$.

Q13. **(a)** Find the equation of line joining the points (1, 2) and (3, 6) using determinants.

(b) Show that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.

(c) Find the value of x, if area of a Δ is $35\,sq.units$ with the vertices $(x,4),(2,-6)$ and $(5,4)$.

Q14. If A, B, C are angles of a triangle, find the value of $\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$.

Q15. Find A(*adj*.A) without finding *adj*.A if $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

Q16. Without actually expanding, evaluate the determinants given below:

a) $\begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$ **b)** $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$ **c)** $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

d) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ **e)** $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$ **f)** $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

Q17. Evaluate the followings using properties of determinants:

a) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ **b)** $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$ **c)** $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

d) $\begin{vmatrix} x+a & a & a \\ b & x+b & b \\ c & c & x+c \end{vmatrix}$ **e)** $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$ **f)** $\begin{vmatrix} 0 & -b & c \\ b & 0 & a \\ -c & -a & 0 \end{vmatrix}$.

g) $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$ **h)** $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

SHORT & LONG ANSWER TYPE QUESTIONS

Q01. Prove that $\begin{vmatrix} x & \sin\delta & \cos\delta \\ -\sin\delta & -x & 1 \\ \cos\delta & 1 & x \end{vmatrix}$ is independent of δ.

Q02. Prove that : $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$.

Q03. Prove the followings:

a) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} = 0$

b) $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

c) $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

d) $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$

e) $\begin{vmatrix} 1 & \alpha & \alpha^2+\beta\gamma \\ 1 & \beta & \beta^2+\alpha\gamma \\ 1 & \gamma & \gamma^2+\alpha\beta \end{vmatrix} = 2(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

f) $\begin{vmatrix} x+y & z+x & y+z \\ y & z & x \\ z & x & y \end{vmatrix} = -(x+y+z)(z-x)^2$

g) $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix} = 5\sqrt{3}\ [\sqrt{6}-5]$

h) $\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{a^2+c^2}{b} \end{vmatrix} = 4abc$

i) $\begin{vmatrix} 2ab & a^2 & b^2 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix} = -(a^3+b^3)^2$

j) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

k) $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

l) $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$

m) $\begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$

n) $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$

o) $\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$

p) $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

q) $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc-a^3-b^3-c^3$

r) $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

s) $\begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (x-1)^3$

t) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

u) $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2 = (2abc)^2$

v) $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

w) $\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$

x) $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

y) $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$

z) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

aa) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

ab) $\begin{vmatrix} 1+x^2-y^2 & 2xy & -2y \\ 2xy & 1-x^2+y^2 & 2x \\ 2y & -2x & 1-x^2-y^2 \end{vmatrix} = (1+x^2+y^2)^3$

ac) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab$

ad) $\begin{vmatrix} a+b+nc & na-a & nb-b \\ nc-c & b+c+na & nb-b \\ nc-c & na-a & c+a+nb \end{vmatrix} = n(a+b+c)^3$

ae) $\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix} = -8$

af) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

Q04. Prove that: $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$.

Q05. If $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = 0$ then, show that $(1+pxyz)=0$. Assume that $x \neq y \neq z$.

Q06. Using properties of determinants, prove that : $\begin{vmatrix} a & b-c & b+c \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$.

Q07. If a,b,c are all positive and p^{th}, q^{th}, r^{th} terms of a G.P. then, prove that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$.

Q08. **a)** If a,b,c are in arithmetic progression, then prove that $\begin{vmatrix} x+1 & x+3 & x+a \\ x+2 & x+5 & x+b \\ x+3 & x+7 & x+c \end{vmatrix} = 0$.

b) If a,b,c are in A.P. then, find the value of determinant $\begin{vmatrix} 2p+4 & 5p+7 & 8p+a \\ 3p+5 & 6p+8 & 9p+b \\ 4p+6 & 7p+9 & 10p+c \end{vmatrix}$.

Q09. If a, b, c are positive and unequal then, show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} < 0$ i.e., the value of the given determinant is negative.

Q10. If a,b,c are real numbers, and it is known that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$. Show that either $a+b+c=0$ or, $a=b=c$.

Q11. Solve the equation: $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$.

Q12. Solve: $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$.

Q13. Prove that: **a)** $\begin{vmatrix} a+b & p+q & x+y \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$ **b)** $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Q14. Using properties of determinants, show that:

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2).$$

Q15. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

☞ This determinant is called a **circular determinant** and its value can be directly used in solving problems of objective types!

Q16. If none of a,b,c is zero, then show that $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$.

Q17. Using the properties of determinants, prove that: $\begin{vmatrix} \sqrt{3}+\sqrt{5} & 2\sqrt{7} & \sqrt{7} \\ \sqrt{35}+\sqrt{6} & 7 & \sqrt{14} \\ 5+\sqrt{21} & \sqrt{35} & 7 \end{vmatrix} = 7\sqrt{5}\,[\sqrt{10}-7]$.

Q18. If $a+b+c=0$, solve : $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$.

OR Using properties of determinants, show that, if $\alpha+\beta+\gamma=0$ and $\begin{vmatrix} \alpha-x & \gamma & \beta \\ \gamma & \beta-x & \alpha \\ \beta & \alpha & \gamma-x \end{vmatrix}=0$, then $x=0$ or $x=\pm\sqrt{\dfrac{3[\alpha^2+\beta^2+\gamma^2]}{2}}$.

Q19. Using properties of determinants, prove that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}=0$.

Q20. If x, y, z are the 10th, 13th and 15th terms of a GP, find the value of $\det(A)$ if $A=\begin{bmatrix} \log x & 10 & 1 \\ \log y & 13 & 1 \\ \log z & 15 & 1 \end{bmatrix}$.

Q21. If [.] denotes the greatest integer function, and $-1\le x<0,\ 0\le y<1,\ 1\le z<2$ then, find value of the determinant: $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$.

Q22. Write the value of the determinant: $\begin{vmatrix} 1 & ab & \dfrac{1}{a}+\dfrac{1}{b} \\ 1 & bc & \dfrac{1}{b}+\dfrac{1}{c} \\ 1 & ca & \dfrac{1}{c}+\dfrac{1}{a} \end{vmatrix}$.

Q23. Find the values of x and y if $\begin{vmatrix} x & -3i & 1 \\ y & 1 & i \\ 0 & 2i & -i \end{vmatrix}=6+11i$.

Q24. If $f(x)=\begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, using properties of determinants find the value of $f(2x)-f(x)$.

BASED ON APPLICATION OF MATRICES & DETERMINANTS

LONG ANSWER TYPE QUESTIONS

Q01. Solve the given system of equations for x, y and z :

a) $x+y=4,\ 2x-3y=9$

b) $-x+2y+3z=3,\ 2x+3y-2z=5,\ 3x+y+4z=11$

c) $x+2y+z=7,\ x+3z=11,\ 2x-3y=1$

d) $2x-y+3z=9,\ x+y+z=6,\ x-y+z=2$

e) $\dfrac{2}{x}+\dfrac{3}{y}+\dfrac{10}{z}=4,\ \dfrac{4}{x}-\dfrac{6}{y}+\dfrac{5}{z}=1,\ \dfrac{6}{x}+\dfrac{9}{y}-\dfrac{20}{z}=2;\ \ x,y,z\ne 0$

f) $5x+3y+z=16,\ 2x+y+3z=19,\ x+2y+4z=25$

g) $3x+\dfrac{4}{y}+7xz=14,\ 2x-\dfrac{1}{y}+3xz=4,\ x+\dfrac{2}{y}-3xz=0$

h) $2x+y-z=4, 3x+y-2z=6, x-z=2$

Q02. If $A=\begin{bmatrix}1&2&5\\2&-1&1\\3&4&-1\end{bmatrix}$ then, find A^{-1}. Hence solve the following system of equations: $x+2y+3z=8,\ 2x-y+4z=8,\ 5x+y-z=16$.

Q03. Find the inverse of the matrix $\begin{bmatrix}2&0&-1\\1&2&3\\2&2&-1\end{bmatrix}$. Hence solve the following system of equations: $2x-z=4$, $x+2y+3z=0$, $2x+2y-z=2$.

Q04. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Find these three numbers by using matrix method.

Q05. Given that $A=\begin{bmatrix}2&-3&5\\3&2&-4\\1&1&-2\end{bmatrix}$ then, find A^{-1}. Using A^{-1} solve the following system of equations: $2x-3y+5z=11,\ 3x+2y-4z=-5,\ x+y-2z=-3$.

Q06. If $A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$, $B=\begin{bmatrix}3&1&-1\\1&3&1\\-1&1&3\end{bmatrix}$ then, find AB. Use this to solve the system of equations: $2x-y+z=-1,\ -x+2y-z=4,\ x-y+2z=-3$.

Q07. Use the product $\begin{bmatrix}1&-1&2\\0&2&-3\\3&-2&4\end{bmatrix}\begin{bmatrix}-2&0&1\\9&2&-3\\6&1&-2\end{bmatrix}$ to solve the following system of equations: $x-y+2z=1,\ 2y-3z=1,\ 3x-2y+4z=2$.

Q08. Find the product $\begin{bmatrix}-4&4&4\\-7&1&3\\5&-3&-1\end{bmatrix}\begin{bmatrix}1&-1&1\\1&-2&-2\\2&1&3\end{bmatrix}$. Hence or otherwise solve the system of equations given as: $x-y+z=4,\ x-2y-2z=9,\ 2x+y+3z=1$.

Q09. If $A=\begin{bmatrix}3&-4&2\\2&3&5\\1&0&1\end{bmatrix}$, $B=\begin{bmatrix}3&4&-26\\3&1&-11\\-3&-4&17\end{bmatrix}$ then, find AB. Using this solve the system of equations given as: $3x-4y+2z=-1,\ 2x+3y+5z=7,\ x+z=2$.

Q10. If $A=\begin{bmatrix}-1&2&3\\2&3&1\\3&-2&4\end{bmatrix}$, find A^{-1}. Hence solve the following system of equations: $-x+2y+3z=3,\ 2x+3y-2z=5$, $3x+y+4z=11$.

Q11. Given $A=\begin{bmatrix}3&-2&3\\2&1&-1\\4&-3&2\end{bmatrix}$, find A^{-1}. Hence solve the system of equations: $3x-2y+3z=8,$ $2x+y-z=1,\ 4x-3y+2z=4$.

Q12. Let $A=\begin{bmatrix}1&-1&1\\2&2&0\\3&3&-1\end{bmatrix}$ and $B=\begin{bmatrix}-2&2&-2\\2&-4&2\\0&-6&4\end{bmatrix}$, verify that $BA=-4I$, where I is a unit matrix. Hence solve the given system of equations: $2y-2x-2z=0, 2x-4y+2z=2, -6y+4z=-8$.

Q13. Solve the following system of equations:

a) $2x+3y=5,\ 6x+9y=15$

b) $5x+3y+7z=4,\ 3x+26y+2z=9,\ 7x+2y+10z=5$

c) $x-y+z=3,\ 2x+y-z=2,\ -x-2y+2z=1$

d) $x+y+z=6,\ x+2y+3z=14,\ x+4y+7z=30$

e) $2x+2y-2z=1,\ 4x+4y-z=2,\ 6x+6y+2z=3$

f) $2x-y+3z=5,\ 3x+2y-z=7,\ 4x+5y-5z=9$

g) $x-2y+z=0,\ x+y-z=0,\ 3x+6y-5z=0$

Q14. For city A, the cost of 4kg wheat, 3kg onion and 2kg rice is ₹60. For city B, the cost of 2kg wheat, 4kg onion and 6kg rice is ₹90. Also for city C, the cost of 6kg wheat, 2kg onion and 3kg rice is ₹70. Find the cost of each item per kilogram by using matrices. Also state which city spends more for buying onions? In recent times, general public was affected due to heavy price rise in the onions. What could be reasons for this in your opinion? Can you suggest any measures to be taken to prevent this issue in future?

Q15. Two schools P and Q want to award their selected students on the values of Tolerance, Kindness and Leadership. The school P wants to award ₹x each, ₹y each and ₹z each for the three respective values to its 3, 2 and 1 students respectively with a total award money of ₹2200. School Q wants to spend ₹3100 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as school P). If the total amount of award for one prize on each value is ₹1200, using matrices, find the award money for each value. Apart from the above these three values, suggest one more value which should be considered for award.

Q16. There are three families. First family consists of 2 male members, 4 female members and 3 children. Second family consists of 3 male members, 3 female members and 2 children. Third family consists of 2 male members, 2 female members and 5 children. Male member earns ₹500 per day and spends ₹300 per day. Female member earns ₹400 per day and spends ₹250 per day, child member spends ₹40 per day. Find the money each family saves per day using matrices? What is the necessity of saving in the family?

Q17. Two schools A and B decided to award prizes to their students for three values honesty (x), punctuality (y) and obedience (z). School A decided to award a total of ₹11000 for the three values to 5, 4 and 3 students respectively while school B decided to award ₹10700 for the three values to 4, 3 and 5 students respectively. If all the three prizes together amount to ₹2700, then:

i. Represent the above situation by a matrix equation and form linear equations using matrix multiplication.

ii. Is it possible to solve the system of equations so obtained using matrices?

iii. Which value you prefer to be rewarded most and why?

Q18. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹x each, ₹y each and ₹z each for the three respective values to its 3, 2 and 1 students with a total award money of ₹1000. School Q wants to spend ₹1500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ₹600, using matrices, find the award money for each value.

Apart from the above three values, suggest one more value for awards.

Q19. Mr.Nakul Saini has invested a part of his income in 10% (bond A) and another part of his income in 15% (bond B). His interest during a certain period is ₹4000. Had he invested 20% more in bond A and 10% more in bond B, his interest would have been increased by ₹500 for the same period. Then: (i) Represent the above situation by a matrix equation and form linear equations using matrix multiplication. (ii) Is it possible to solve the system of equations so obtained by matrices? If yes, solve it too.

Q20. For keeping fit, X people believe in morning walk, Y people believe in yoga and Z people join gym. Total no. of people are 70. Further 20%, 30% and 40% people are suffering from any diseases who believe in morning walk, yoga and gym respectively. Total no. of such people is 21. If morning walk costs ₹0, yoga costs ₹500/month and gym costs ₹400/month and total expenditure is ₹23000.

(i) Formulate a matrix problem.

(ii) Calculate the no. of each type of people.

(iii) Why exercise is important for health?

Q21. In a Legislative assembly election, a political party hired a public relation firm to promote its candidate in three ways: telephone, house calls and letters. The numbers of contacts of each type in three cities A, B & C are (500, 1000, 5000), (3000, 1000, 10000) and (2000, 1500, 4000), respectively. The party paid ₹3700, ₹7200, and ₹4300 in cities A, B & C respectively. Find the costs per contact using matrix method. Keeping in mind the economic condition of the country, which way of promotion is better in your view?

Q22. A trust fund has ₹30,000 is to be invested in two different types of bonds. The first bond pays 5% interest per annum which will be given to orphanage and second bond pays 7% interest per annum which will be given to an N.G.O. cancer aid society. Using matrix multiplication, determine how to divide ₹30,000 among two types of Bonds if the trust fund obtains an annual total interest of ₹1800. What are the values reflected in this question?

Q23. A school has to reward the students participating in co-curricular activities (Category I), with 100% attendance (Category II) and brave students (Category III) in a function. The sum of the numbers of all the three category students is 6. If we multiply the number of students of category III by 2 and add to the number of students of category I to the result, we get 7. By adding II and III category students to three times the I category students, we get 12. Form the matrix equation and, hence solve it.

Q24. Two farmers Ramkrishna and Hari Prasad cultivated three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices 'A' and 'B' :

September Sales (in Rupees)

$$A = \begin{pmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{pmatrix} \begin{matrix} \\ \text{Ramkrishna}, \\ \text{Hari Prasad} \end{matrix}$$

October Sales (in Rupees)

$$B = \begin{pmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{pmatrix} \begin{matrix} \\ \text{Ramkrishna} \\ \text{Hari Prasad} \end{matrix}$$

(i) Find the combined sale in September and October for each farmer in each variety.
(ii) Find the decrease in sales from September to October.
(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.
(iv) Which farmer gets more profit in the overall sales for both the months?
(v) Which farmer in your opinion is more resourceful and why?

Q25. A total amount of ₹7000 Is deposited in three different savings bank accounts with annual interest rates of 5%, 8% and $8\frac{1}{2}\%$ respectively. The total annual interest from these three accounts is ₹550. Equal amounts have been deposited in 5% and 8% savings accounts. Find the amount deposited in each of the three accounts, with the help of matrices.